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General Properties of the Liouville Operator

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We study the self-adjointness of the Liouvillian of a symmetric operator. We also discuss some cases of the spectrum of the Liouville operator of a self-adjoint Hamiltonian with purely continuous singular spectrum. The presence of an absolutely continuous part for the spectrum of Liouvillians corresponding to Hamiltonians with purely continuous singular spectrum shows that quantum theory in Hilbert and Liouville spaces is not equivalent.

1. INTRODUCTION

In the present paper, we study certain general properties of the quantum Liouville operator. The Liouville operator plays an important role in classical and quantum statistical mechanics, since it provides the law of evolution of states and observables (Prigogine, 1962; Reichl, 1980). In both cases, there exists a formal relation between the Liouville operator $\mathbb{L}$ and its corresponding Hamiltonian H, which is given by $\mathbb{L}\rho = \{H, \rho\}$, where ρ denotes the state. Here, the brackets denote the Poisson brackets for the classical case and the commutator times $i\hbar$ in the quantum case.

It is well known that the Liouville operator $\mathbb{L}$ (also called the Liouvillian) corresponding to an essentially self-adjoint Hamiltonian H is also e.s.a. (Spohn, 1976; Reed and Simon, 1972). Moreover, the spectrum of $\mathbb{L}$ is given by the spectrum of H according to the following formula:

$$\sigma(\mathbb{L}) = \sigma(H) - \sigma(H) \tag{1}$$

(In this paper, $\sigma(A)$ will represent the spectrum of the operator A.) Also, it is obvious that the set of eigenvalues of $\mathbb{L}$ is given by all the differences

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between eigenvalues of H. Spohn (1976) gave a formula to obtain the continuous and singular spectrum of $\mathbb{L}$, given the eigenvalues as well as the continuous and singular spectrum of H. Part of the contents of the present paper are motivated by Spohn (1976).

This paper is divided into two sections. In the first part, we present some extra results concerning the formal Liouvillian of a symmetric (e.s.a. or not) operator. First of all, it seems natural that different self-adjoint extensions of the same symmetric non-e.s.a. operator have different Liouvillians. We start the next section by giving a formal proof of this fact, after giving a definition of the formal Liouvillian for every symmetric operator. We prove even more; we give a necessary and sufficient condition for the Liouvillians of two self-adjoint operators to be identical. In addition, we show that the formal Liouvillian of a symmetric operator A always has self-adjoint extensions, even if A does not have self-adjoint extensions.

The last section is devoted to some important comments on the singular spectrum of the Liouvillian of a self-adjoint operator. This has been motivated by Spohn (1976) since, the way in which the author constructs the singular spectrum of the Liouvillian of a given Hamiltonian is wrong. Therefore, we start this second part by showing that the arguments given in his proof are not correct. This fosters further investigations which may have dramatic consequences in the interpretation of quantum mechanics in Liouville space.

The key point is that the complex difference $\sigma_{sc}(A) - \sigma_{sc}(A)$ of the singular spectrum of a self-adjoint operator A does not need to be a singular set of zero Lebesgue measure. The continuous singular spectrum of an operator A is characterized by the fact that the measure $\langle\psi|E(x)\psi\rangle$, where $E(x)$ is the spectral measure of A, is continuous singular with respect to the Lebesgue measure whenever $\psi \in \mathcal{H}_{sc}$, the continuous singular subspace of the Hilbert space. One may find situations in which a self adjoint operator has only a singular continuous spectrum of the Cantor form, but the spectrum of the corresponding Liouvillian is an interval of the real line. Another striking possibility is that the absolutely continuous spectrum of the Liouvillian of a self-adjoint operator A may be contained in the set $\sigma_{sc}(A) - \sigma_{sc}(A)$. These results have fostered research concerning the relation between the spectrum of an operator and the spectrum of its corresponding Liouvillian.

Bas and Pavlov (1995) found an example of a Hamiltonian with purely singular spectrum for which the spectrum of its Liouvillian has an absolutely continuous component. Here, we generalize this result in the sense that it is always possible to find a Hamiltonian with purely continuous singular spectrum concentrated on a set of arbitrarily small Hausdorff dimension for which the spectrum of the Liouvillian has an absolutely continuous component. We make the proof of this statement in the language of spectral measures (Berezanski, 1968; Dalecki and Krein, 1965; Iorio, 1978).

The result by Bas and Pavlov and ours show their importance in the context of scattering theory. It is a general belief that scattering theory is equivalent on both Hilbert space and Liouville space (Spohn, 1976; Prugovecki, 1981). However, one can see that it is always possible to construct Hamiltonians for which no scattering states may exist, because of the absence of absolutely continuous spectrum, and nonetheless their corresponding Liouvillians indeed have scattering states. This fact shows the nonequivalence between scattering on Hilbert and on Liouville spaces, contrary to what is usually accepted (Prugovecki, 1981).

2. SELF-ADJOINTNESS OF THE FORMAL LIOUVILLIAN

Definition. Let A be a symmetric operator. We define its formal Liouvillian as the closed symmetric operator defined as $\mathbb{L}_A = A \otimes I - I \otimes A$ on the Hilbert space $\mathcal{H} \otimes \mathcal{H}^\times$, where $\mathcal{H}^\times$ denotes the dual space of $\mathcal{H}$ (usually identified with $\mathcal{H}$) and I is the identity operator on $\mathcal{H}$.

Next, we present two interesting results concerning Liouvillians. The former is very intuitive, but it seems to confirm the equivalence between Hilbert and Liouville spaces.

Proposition. Let A and B be two self-adjoint operators on $\mathcal{H}$. Let $\mathbb{L}_A$ and $\mathbb{L}_B$ be their respective Liouvillians. Then, $\mathbb{L}_A = \mathbb{L}_B$ if and only if $A = B + \alpha I$, where α is a real constant.

Proof. Let ψ and φ be two arbitrary elements in $\mathcal{H}$. Then,

$$e^{-it\mathbb{L}_A}\psi \otimes \varphi = e^{-itA}\psi \otimes e^{+itA}\varphi \text{ and } e^{-it\mathbb{L}_B}\psi \otimes \varphi = e^{-itB}\psi \otimes e^{+itB}\varphi \tag{2}$$

Now, assume that the two vectors in (2) are the same for any given pair ψ, φ. Then, one has (Weidmann, 1980)

$$e^{-itA}\psi = c(t)e^{-itB}\psi\ ; \qquad e^{+itA}\varphi = c(t)^{-1}e^{+itB}\varphi \tag{3}$$

A very important point is that $c(t)$ must be the same for every vector in $\mathcal{H}$ and for a given value of t. To see this, we keep ψ fixed and let φ be arbitrary in $\mathcal{H}^\times$.

Due to the arbitrariness of ψ and φ, one has

$$e^{-itA} = c(t)e^{-itB} \tag{4}$$

From (4), one immediately gets that $c(t) = e^{i\alpha(t)}$, where $\alpha(t)$ is a real function of $\mathbb{R}$. Using this form for $c(t)$ in (4) with $e^{-i(t+\tau)A}$, one easily gets that

$$e^{i\alpha(t+\tau)} = e^{i(\alpha(t)+\alpha(\tau))} \tag{5}$$

Consider now

$$U(t) := e^{i\alpha(t)} = e^{-itA} e^{itB} \tag{6}$$

With the aid of (5) one immediately shows that $U(t)$ is a unitary group with parameter t. It is also strongly continuous with respect to the variable t on $\mathcal{H}$. By the Stone theorem, there must exist a self-adjoint operator S such that

$$U(t) = e^{iSt} = e^{i\alpha(t)} \tag{7}$$

Let $\mathcal{D}(S)$ be the domain of S. Let $\psi \in \mathcal{D}(S)$. Then, the following limit exists:

$$\lim_{\tau\to 0} \frac{U(\tau) - I}{\tau} \psi = iS\psi = \lim_{\tau\to 0} \frac{e^{i\alpha(\tau)} - 1}{\tau} \psi \tag{8}$$

Therefore,

$$\lim_{\tau\to 0} \frac{e^{i\alpha(\tau)} - 1}{\tau} \tag{9}$$

exists. This limit is a complex number, which we call $i\alpha$. Therefore, $iS\psi = i\alpha\psi$ for all $\psi \in \mathcal{D}(S)$. Since $\mathcal{D}(S)$ is dense in $\mathcal{H}$, one has that $S\psi = \alpha\psi$ for all $\psi \in \mathcal{H} \Rightarrow S = \alpha I$. Since S is self-adjoint, α is real. Then, one has

$$e^{-itA} = e^{i\alpha t I} e^{-itB} = e^{-it(B-\alpha I)} \Rightarrow A = B - \alpha I \tag{10}$$

The converse is obvious. ■

As a corollary, one can answer the following question: Let A be a symmetric, densely defined operator with equal deficiency indices different from zero. Do its self-adjoint extensions have the same Liouvillian? After the previous theorem, one sees that the answer is negative. The proof of this statement is trivial.

Our next question is the following: Being given a symmetric operator on $\mathcal{H}$ having nonequal deficiency indices (and therefore without self-adjoint extensions), what one can say about the deficiency indices of its Liouvillian? The answer is given by the next result.

Proposition. Let H be a maximal symmetric operator with different deficiency indices. Then its corresponding Liouvillian has both deficiency indices equal to ∞.

Proof. It is a consequence of the following result (Dunford and Schwartz, 1963). Let T be a maximal symmetric operator on $\mathcal{H}$. Then $\mathcal{H}$ can be decomposed into an orthogonal direct sum:

$$\mathcal{H} = \sum_{n=0}^{N} \mathcal{H}_n; \qquad N \text{ finite or infinite} \tag{11}$$

of subspaces invariant under T and $T^\dagger$, such that:

(i) The restriction T_0 of T to $\mathcal{H}_0$ is self-adjoint.

(ii) For $n \neq 0$, there exists a unitary operator U_n from $\mathcal{H}_n$ onto $L^2(\mathbb{R}^+)$ such that if T_n denotes the restriction of T to $\mathcal{H}_n$, the operator $U_n T_n U_n^{-1}$ is the operator $\pm iD$, defined as
$\mathcal{D}(D) = \{f \in L^2(\mathbb{R}^+) \,/\, f(x)$ is absolutely continuous,

$$f'(x) \in L^2(\mathbb{R}^+); f(0) = 0\} \quad Df(x) = f'(x) = \frac{d}{dx} f(x) \tag{12}$$

We have $+iD$ for all n if the positive (n_+) deficiency index of T is zero and $-iD$ for all n if the negative deficiency index of T is zero.

Now, consider the decomposition of $\mathcal{H}$ given in (11). Take $n \neq 0$. Consider the orthogonal projection P_n: $\mathcal{H} \mapsto \mathcal{H}_n$ and $\mathcal{P}_n = P_n \otimes P_n$. Now, define

$$\begin{aligned} \mathbb{L}_n &= \mathcal{P}_n \mathbb{L} \mathcal{P}_n = P_n \otimes P_n(H \otimes I - I \otimes H)\, P_n \otimes P_n \\ &= (P_n H P_n) \otimes P_n - P_n \otimes (P_n H P_n) \end{aligned} \tag{13}$$

Since H leaves $\mathcal{H}_n$ invariant, $P_n H P_n$ is the restriction of H to $\mathcal{H}_n$. We know that this restriction is unitarily equivalent to either $+iD$ or $-iD$ on $L^2(\mathbb{R}^+)$. Thus, $\mathbb{L}_n$ is unitarily equivalent to $K_n = \pm i(D \otimes I - I \otimes D)$. Since (Weidman, 1980) $K_n^\dagger > D^\dagger \otimes I - I \otimes D^\dagger$, the equations $(K_n^\dagger \pm iI)\psi = 0$ have at least the following solutions:

$$\psi_\alpha(x, y) = A e^{\alpha x} e^{\alpha y} e^{\mp iy} \tag{14}$$

where $\alpha < 0$. These solutions belong to $L^2(\mathbb{R}^+ \times \mathbb{R}^+)$ (where $\mathbb{R}^+ = [0,\infty)$). Since solutions with different values of α are linearly independent, one concludes that the deficiency indices of k_n are both infinite. Therefore, we conclude that the deficiency indices of $\mathbb{L}_n$ are both infinite. Since $\mathbb{L}$ is (the closure of) the orthogonal sum of the $\mathbb{L}_n$, this implies that $\mathbb{L}$ has both deficiency indices equal to ∞. ■

Remark. Since we define the Liouville space as $\mathcal{H} \otimes \mathcal{H}^\times$, one may be tempted to define the Liouvillian of a given symmetric operator H as the closure of $\mathbb{L} = H \otimes I - I \otimes H^\dagger$. However, if H were not essentially self-adjoint, then $\mathbb{L}$ would not be even symmetric. To prove it, take $\psi_1, \psi_2 \in \mathcal{D}(H)$ and $\varphi_1, \varphi_2 \in \mathcal{D}(H^\dagger)$, all different from zero, and assume that $\mathbb{L}$ is symmetric. Then, one has

$$\begin{aligned} &\langle H\psi_1 \otimes \varphi_1 - \psi_1 \otimes H^\dagger \varphi_1 | \psi_2 \otimes \varphi_2 \rangle \\ &- \langle \psi_1 \otimes \varphi_1 | H\psi_2 \otimes \varphi_2 - \psi_2 \otimes H^\dagger \varphi_2 \rangle = 0 \end{aligned} \tag{15}$$

This implies that

$$\langle H\psi_1 \otimes \varphi_1 | \psi_2 \otimes \varphi_2\rangle - \langle \psi_1 \otimes H^\dagger \varphi_1 | \psi_2 \otimes \varphi_2\rangle$$
$$= \langle \psi_1 \otimes \varphi_1 | H\psi_2 \otimes \varphi_2\rangle - \langle \psi_1 \otimes \varphi_1 | \psi_2 \otimes H^\dagger \varphi_2\rangle \tag{16}$$

or

$$\langle H\psi_1 | \psi_2\rangle \langle \varphi_1 | \varphi_2\rangle - \langle \psi_1 | \psi_2\rangle \langle H^\dagger \varphi_1 | \varphi_2\rangle = \langle \psi_1 | H\psi_2\rangle \langle \varphi_1 | \varphi_2\rangle$$
$$- \langle \psi_1 | \psi_2\rangle \langle \varphi_1 | H^\dagger \varphi_2\rangle \tag{17}$$

Since H is symmetric, this equation shows that $H^\dagger$ is also symmetric and hence that H is essentially self-adjoint, which is a contradiction. ■

3. CANTOR SPECTRUM

We would like to consider some interesting, even paradoxical, properties of the spectrum of the Liouvillian corresponding to an operator with a singular continuous spectrum. The additional motivation for these consideration is a partially incorrect paper of Spohn (1976) in which the author describes connections between different parts of the spectrum of a Hamiltonian, i.e., absolutely continuous, singular continuous, and point spectrum, and the corresponding spectra of the Liouvillian.

The interest of this discussion goes beyond the Spohn result. One of its consequences has been to show the nonequivalence between the Hilbert space and the Liouville space descriptions of quantum mechanics, as mentioned in the Introduction. This may have dramatic consequences from the point of view of the scattering theory and resonance behavior that are under present investigation.

In Spohn (1976) it is proved that $\sigma(\mathbb{L}) = \sigma(H) - \sigma(H)$, where $\sigma(A)$ stands for the spectrum of the operator A. This result is not new, since a more general one was already known (Reed and Simon, 1972). Moreover, the same paper gives a similar characterization for absolutely continuous, singular continuous, and point spectra. What is wrong there is the proof that $\sigma(\mathbb{L})_{sc} = \sigma(H)_{sc} - \sigma(H)_{sc}$. The argument used there is that if $\sigma(H)_{sc}$ is the set of Lebesgue measure 0 (where sc stands for singular continuous), then the complex difference $\sigma(H)_{sc} - \sigma(H)_{sc}$ is also a set of the Lebesgue measure 0. This is, however, not true, as we shall show below.

To see it, let us consider a Hamiltonian operator H for which $\sigma(H)$ is a Cantor set. For example, there is an absolutely summable sequence $\{a_n\}$ such that the spectrum of the Hamiltonian

$$H = -\frac{d^2}{dx^2} + \sum_{n=0}^{\infty} a_n \cos(x2^{-n}) \tag{18}$$

is the Cantor set (Bellisard, 1982, and references therein). Suppose that $\sigma(H) = C$, where C is the Cantor set on the interval [0, 1]. According to the general property of spectra of functions of the Hamiltonian (Reed and Simon, 1972) we have formula (1).

However, according to a theorem of Steinhaus (1917) we have

$$C - C = [-1, 1] \tag{19}$$

i.e., the complex difference of two Cantor sets covers the **whole** interval and is thus the set of nonzero Lebesgue measure, contrary to what is claimed in Spohn (1976). This property of the Cantor set does not, however, imply that the spectrum of the Liouvillian is absolutely continuous. To show this, let us consider the devil's staircase distribution function $F(x)$ on the Cantor set C, which has constant value equal to $k/2^n$ (k is such that the fraction $k/2^n$ is nonreducible) on each interval which is removed in the nth step of the construction of the Cantor set (Billingsley, 1985). [The devil's staircase is defined as follows: it is 1/2 on the interval (1/3,2/3), 1/4 on (1/9,2/9), 3/4 on (7/9,8/9), 1/8 on (1/27,2/27), 3/8 on (7/27, 8/27), 5/8 on (19/27, 20/27), 7/8 on (25/27, 26/27) and so on.]

It is well known and not difficult to check that $F(x)$ is a nondecreasing continuous function such that $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$. We can show that $F(x)$ is a spectral measure corresponding to a Hamiltonian. Precisely speaking, the spectral theory for self-adjoint operators shows that there exists a self-adjoint operator H on a Hilbert space $\mathcal{H}$ with the spectral resolution

$$H = \int_{\sigma(H)} \lambda \, dE_\lambda \tag{20}$$

and an element $h \in \mathcal{H}$ such that

$$F(x) = \langle E_x h, h \rangle \qquad \text{for} \qquad x \in \mathbb{R} \tag{21}$$

Consider the Liouvillian, corresponding to this H which is defined on the Hilbert tensor product $\mathcal{H} \otimes \mathcal{H}^\times$ and is given by

$$\mathbb{L} = H \otimes I - I \otimes H \tag{22}$$

$$\mathbb{L} = \int_{\sigma(H)-\sigma(H)} \lambda \, d\mathbb{E}_\lambda \tag{23}$$

Then, one can show that $\mathbb{E}_\lambda$ can be represented as (see Appendix)

$$\mathbb{E}_\lambda = \int_{-\infty}^{\infty} (E_{\lambda+\mu} \otimes I) \, d(I \otimes E_\mu) \tag{24}$$

In particular,

$$\langle \mathbb{E}_\lambda h \otimes h, h \otimes h \rangle = \int_{-\infty}^{\infty} \langle E_{\lambda+\mu} h, h \rangle = \int_{-\infty}^{\infty} F(\lambda + \mu)\, dF(\mu) \quad (25)$$

This implies that the function $x \mapsto G(-x)$, where

$$G(x) := \langle \mathbb{E}_x h \otimes h, h \otimes h \rangle \quad (26)$$

is a convolution of the function $x \mapsto F(-x)$ with $x \mapsto F(x)$. By general properties of the convolution the function, $G(x)$ is continuous. We shall show that it is, however, not absolutely continuous. In order to show this, let us consider the Fourier–Stieltjes transform

$$\hat{F}(t) = \int_0^1 e^{itx}\, dF(x) \quad (27)$$

Let us find the explicit expression for $\hat{F}(t)$:

$$\begin{aligned} \hat{F}(t) &= \int_0^1 e^{itx} dF(x) = \frac{1}{2} \int_0^{1/3} e^{itx} dF(3x) + \frac{1}{2} \int_{2/3}^1 e^{itx} dF(3x - 2) \\ &= \frac{1}{2} \int_0^1 e^{itx/3} dF(x) + \frac{1}{2} \int_0^1 e^{itx/3}\, e^{it2/3}\, dF(x) \\ &= \frac{1}{2} [1 + e^{2it/3}]\, \hat{F}\left(\frac{t}{3}\right) \\ &= \frac{1}{2} [1 + e^{2it/3}]\, \frac{1}{2} [1 + e^{2it/3^2}]\, \hat{F}\left(\frac{t}{3^2}\right) \\ &= \left(\prod_{k=1}^{\infty} \frac{1}{2^k} [1 + e^{2it/3}] \right) \hat{F}(0) \end{aligned} \quad (28)$$

On the first identity in the second row, we have used the fact that $F(x)$ satisfies the following functional equation of the De-Rham type (compare to Tasaki *et al.*, 1993

$$F(x) = \begin{cases} \frac{1}{2} F(3x), & 0 \le x < \frac{1}{3} \\ \frac{1}{2}, & \frac{1}{3} \le x < \frac{2}{3} \\ \frac{1}{2} F(3x - 2) + \frac{1}{2}, & \frac{2}{3} \le x \le 1 \end{cases} \quad (29)$$

One easily finds that $\hat{F}(0) = 1$. Then we get the desired formula for $\hat{F}(t)$:

$$\hat{F}(t) = \left(\prod_{k=1}^{\infty} [1 + e^{2it/3}] \right) \tag{30}$$

Choosing $t_n = \pi 3^n$, we see that

$$\hat{F}(t_n) = \prod_{k=n+1}^{\infty} \frac{1}{2} [1 + e^{2\pi i/(3k^{-n})}] = \prod_{k=1}^{\infty} \frac{1}{2} [1 + e^{2\pi i/3k} \tag{31}$$

does not depend on n. Moreover, $\hat{F}(t_n)$ is nonzero. Indeed, taking n_0 such that $\pi/3^n < 1/2^n$, for $n > n_0$, and using the equality $\prod_{n=1}^{\infty} \cos(x/2^n) = (\sin x)/x$, we have

$$|\hat{F}(t_n)| = \prod_{k=1}^{\infty} \cos\frac{\pi}{3^k} > \prod_{k=1}^{n_0} \cos \frac{\pi}{3^k} \prod_{k=n_0+1}^{\infty} \cos \frac{1}{2^k}$$

$$= \prod_{k=1}^{n_0} \cos \frac{\pi}{3^k} \prod_{k=1}^{\infty} \cos \frac{2^{-n_0}}{2^k} = 2^{n_0} \sin \frac{1}{2^{n_0}} \prod_{k=1}^{n_0} \cos \frac{\pi}{3^k} > 0 \tag{32}$$

This means that the Fourier transform $\hat{F}(t)$ does not converge to zero as $t \to \infty$. The same is true for the Fourier transform $\hat{G}(t)$ of $G(x)$, because $|\hat{G}(t)| = |\hat{F}(t)|^2$. Therefore, in view of the Riemann Lebesgue lemma, $F(x)$ as well as $G(x)$ cannot be absolutely continuous.

In fact more can be shown, namely that $G(t)$ is a strictly monotonically increasing and continuous function on the interval $[-1,1]$, although not differentiable at any point.

The above example shows that the property of the spectrum being singular should be interpreted as the property of spectral measures rather than sets. This means that the spectrum of the Liouvillian need not be a thin set in the sense of being a set of Lebesgue measure zero, even if this is true for the Hamiltonian. To illustrate this point, let us consider a Hamiltonian constructed in the following way: let A be a self-adjoint operator on a Hilbert space $\mathcal{H}_1$, like the one considered above, having as (singular) spectrum the Cantor set in the interval [0, 1], and let B be a self-adjoint operator on a Hilbert space $\mathcal{H}_2$ having absolutely continuous spectrum $[1/3 + \varepsilon, 2/3 - \varepsilon]$, $\varepsilon > 0$. Then consider the operator $H = A \otimes I + I \otimes B$ defined on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. After Reed and Simon (1972) one gets

$$\sigma(H) = \sigma(A \otimes I) + \sigma(I \otimes B) = \sigma(A) + \sigma(B) \tag{33}$$

Therefore the spectrum of H consists of two separate parts. Its singular part has the Lebesgue measure zero and its absolutely continuous part has a positive measure. It turns out, however, that going to the corresponding Liouvillian, we obtain that the continuous part of the spectrum is a proper subset of the set $\sigma_{sc}(H) - \sigma_{sc}(H)$.

The example that we have discussed in this section does not provide us, however, with further information. For instance, one may ask whether in this kind of situation the spectrum of the Liouvillian has an absolutely continuous component so that its spectral measure is the sum of singular measures plus absolutely continuous measures with respect to the Lebesgue measure. With this idea in mind, Bas and Pavlov (1995) found an example of a Hamiltonian with purely singular continuous spectrum having an absolutely continuous component. Their construction is based on a paper by Levenberg *et al.* showing that the convolution of two singular measures may yield the Lebesgue measure on an interval of the real line.

Now, based on Levenberg *et al.* (1988), we shall show a stronger result. Namely, we shall show that it is always possible to find a Hamiltonian which has the purely singular spectral measure concentrated on a set of an arbitrary small Hausdorff dimension.

The key point in the further construction is a result of which Levenberg *et al.* (1988) shows that for a given Hausdorff measure χ one can construct two subsets E_1 and E_2 of the unit interval $I = [0,1]$ such that

$$E_1 + E_2 = \{x + y / x \in E_1; y \in E_2\} = I \tag{34}$$

and

$$\chi(E_1) = \chi(E_2) = 0 \tag{35}$$

Moreover, the *natural* probability measures μ_i supported on E_i, $i = 1, 2$, have the property that the convolution $\mu_1 * \mu_2$ is Lebesgue on I.

The term natural probability measure used above requires an explanation. The sets E_i, $i = 1, 2$, constructed in Levenberg *et al.* (1988) are nowhere-dense Cantor-type sets, and therefore sets of zero Lebesgue measure, constructed by removing consecutively some number of intervals. The whole procedure is infinite, but on each step n we have sets $E_{i,n}$ consisting of a finite number of disjoint closed intervals of the same length. The corresponding natural probability measure $\mu_{i,n}$ ($i = 1, 2$) is defined as a measure on the Borel subsets of I through its density

$$d\mu_{i,n}(x) = \frac{1}{|E_{i,n}|} \mathbf{1}_{E_{i,n}}(x)\, dx \tag{36}$$

where $|E_{i,n}|$ is the total length of all subintervals of $E_{i,n}$ and $\mathbf{1}_{Ei,n}{}^{(x)}$ its characteristic function (indicator). Then μ_i are defined as limits of $\mu_{i,n}$ as $n \to \infty$. Of course the measures μ_i no longer have densities; they are both singular with respect to the Lebesgue measure on I.

We should show that one can construct a Hamiltonian with a given singular spectral measure. However, before doing it, let us introduce some important facts concerning convolution of measures.

Let μ_1, μ_2 be two real measures, $\mu_1 \otimes \mu_2$ its product, and $\mu_1 * \mu_2$ its convolution. Recall that we define $\mu_1 * \mu_2$ as the composition of the product $\mu_1 \otimes \mu_2$ with the measurable map s: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$s(x,y) = x + y \tag{37}$$

Now, we make another construction that we call the *anticonvolution* of μ_1 and μ_2, denoted by $\mu_1 \underline{*} \mu_2$. This is the composition of the measure $\mu_1 \otimes \mu_2$ with the measurable map d: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$d(x,y) = x - y \tag{38}$$

Reflection of a Measure. Being given a real measure μ, we define its *reflection* as the composition of the measurable map r: $\mathbb{R} \to \mathbb{R}$

$$r(x) = -x \tag{39}$$

with μ, so that

$$\bar{\mu} = \mu \circ r \tag{40}$$

Then, we have $\bar{\mu}(A) = \mu(-A)$.

Lemma. The reflection has the following properties:
(i) $\overline{\mu_1 + \mu_2} = \bar{\mu}_1 + \bar{\mu}_2$.
(ii) $\overline{(\bar{\mu}_1)} = \mu$.
(iii) $\mu_1 * \mu_2 = \mu_1 * \bar{\mu}_2$.

Proof. The proofs for (i) and (ii) are obvious. In order to show (iii), let us define the (measurable) transformation t: $\mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$

$$t(x, y) = (x, -y) \tag{41}$$

We want to show first that

$$\mu_1 \otimes \bar{\mu}_2 = (\mu_1 \otimes \mu_2) \circ t \tag{42}$$

For that, we need only to show this identity for sets of the form $A \times B$, where A and B are measurable sets. Indeed

$$
\begin{aligned}
((\mu_1 \otimes \mu_2) \circ t)(A \times B) &= (\mu_1 \otimes \mu_2)t^{-1}(A \times B) \\
&= (\mu_1 \otimes \mu_2)(A \times (-B)) \\
&= \mu_1(A)\mu_2(-B) = \mu_1(A)\bar{\mu}_2(B) \\
&= \mu_1 \otimes \bar{\mu}_2(A \times B)
\end{aligned} \tag{43}
$$

Therefore, taking any measurable set C, we have

$$
\begin{aligned}
\mu_1 \,\bar{*}\, \mu_2(C) &= (\mu_1 \otimes \mu_2)d^{-1}(C) \\
&= \iint_{\{(x,y):x-y\in C\}} d\mu_1 \otimes \mu_2 \\
&= \iint_{t^{-1}\{(x,y):x+y\in C\}} d\mu_1 \otimes \mu_2 \\
&= \iint_{\{(x,y):x+y\in C\}} d\mu_1 \otimes \bar{\mu}_2 \\
&= \mu_1 \otimes \bar{\mu}_2(s^{-1}(C)) \\
&= \mu_1 * \bar{\mu}_2(C) \quad \blacksquare
\end{aligned} \tag{44}
$$

Now, take μ_1 and μ_2 two arbitrary singular measures on I such that $\mu_1 * \mu_2$ is the Lebesgue measure on the same interval. We know that these kinds of objects exist due to the above result of Levenberg *et al.* (1988). Define $\mu = \mu_1 + \bar{\mu}_2$, which is a singular measure with support $E_1 \cup E_2$.

Now, it is easy to find a Hamiltonian with the desired spectral properties. The simplest example of an operator with spectrum $E_1 \cup E_2$ and μ as spectral measure is given by the multiplication operator

$$
Hf(x) = xf(x) \tag{45}
$$

on the Hilbert space $L^2(I, \mu)$. Its spectral resolution

$$
H = \int_{\sigma(H)} \lambda \, dE_\lambda
$$

can be given explicitly if we define projectors E_λ as the operators of multiplication by indicators $\mathbf{1}_{[0,\lambda]}$ (projectors on the spaces L^2 ([0, λ], μ)). Then $e \equiv 1$ is the corresponding cyclic vector and $\mu([0, \lambda]) = \langle E_\lambda e, e\rangle$.

The Liouvillian corresponding to H has the spectral projectors $\mathbb{E}_\lambda$, and the measure generated by $(\mathbb{E}_\lambda e \otimes e, e \otimes e)$ is given by the convolution of μ and $\bar{\mu}$. We have

$$
\begin{aligned}
\mu * \bar{\mu} &= (\mu_1 + \bar{\mu}_2) * \overline{(\mu_1 + \bar{\mu}_2)} \\
&= (\mu_1 + \bar{\mu}_2) * (\bar{\mu}_1 + \mu_2)
\end{aligned}
$$

$$= \mu_1 * \bar{\mu}_2 + \mu_1 * \mu_2 + \bar{\mu}_2 * \bar{\mu}_1 + \bar{\mu}_2 * \mu_2$$

Here, $\mu_1 * \mu_2$ is an absolutely continuous measure by hypothesis. Because $\bar{\mu}_2 * \bar{\mu}_1 = \bar{\mu}_1 * \bar{\mu}_2$ and $\bar{\mu}_1$, $\bar{\mu}_2$ are reflections of $\bar{\mu}_1$, $\bar{\mu}_2$, then $\bar{\mu}_2 * \bar{\mu}_1$ is also absolutely continuous. These two convolutions form the absolutely continuous part of the spectral measure of the Liouvillian.

Thus, we have constructed a Hamiltonian with continuous singular spectrum supported on the nowhere-dense set (the interior of its closure is the empty set) $E_1 \cup E_2$ such that its Liouvillian has a nonempty, absolutely continuous spectrum. This result justifies our claim that quantum theory on Hilbert and Liouville spaces is not equivalent.

APPENDIX

In this Appendix, we wish to present the proof of formula (24), which relates the spectral decomposition $\mathbb{E}_\lambda$ of $\mathbb{L}$ to the spectral decomposition E_α of H, where $\mathbb{L} = H \otimes I - I \otimes H$. To do it, we use (2) in the following form (Weidmann, 1980):

$$e^{i\mathbb{L}t} = e^{iHt} \otimes e^{-iHt} \tag{A.1}$$

Let h, g be vectors in the Hilbert space $\mathcal{H}$. Then (A.1) implies that

$$(e^{i\mathbb{L}t} h \otimes g, h \otimes g) = \int_{-\infty}^{\infty} e^{i\lambda t}\, d(\mathbb{L}_\lambda h \otimes g,\ h \otimes g) \tag{A.2}$$

and

$$\begin{aligned}
&(e^{iHt} \otimes e^{-iHt}(h \otimes g),\ h \otimes g)) \\
&\quad = (e^{iHt}h,\ h)(e^{-iHt}g,\ g) \\
&\quad = \left[\int_{-\infty}^{\infty} e^{i\alpha t}\, d(E_\alpha h,\ h)\right]\left[\int_{-\infty}^{\infty} e^{-i\beta t}\, d(E_\beta g,\ g)\right] \\
&\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha-\beta)t}\, d(E_\alpha h,\ h)\, d(E_\beta g,\ g)
\end{aligned} \tag{A.3}$$

Writing $\alpha - \beta = \lambda$, we see that (A.3) is equal to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\lambda t}\, d(E_{\lambda+\beta}\, h,h)\, d(E_\beta\, g,g) \tag{A.4}$$

If the Fourier transform of two measures on $\mathbb{R}$ coincide, these measures must be equal. Thus,

$$(\mathbb{E}_\lambda\, h \otimes g,\ h \otimes g = \int_{-\infty}^{\infty} (E_{\lambda+\beta}\, h,h)\ d(E_\beta\ \mathrm{g,g}) \qquad \text{(A.5)}$$

This formula can be written analogously as

$$\mathbb{E}_\lambda = \int_{-\infty}^{\infty} (E_{\lambda+\beta} \otimes I)\ d(I \otimes E_\beta) \qquad \text{(A.6)}$$

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